

RANDOM CONSERVATIVE CYLINDER FLOWS

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ABSTRACT. In the first part of this work, we study ergodic properties of random iterations of conservative cylinder flows, given by \mathbb{R} -extensions of an ergodic probability measure-preserving transformation, and obtain a version of Kakutani's random ergodic theorem for such systems. In the second part, we restrict attention to the case of two \mathbb{R} -extensions of an uniquely ergodic probability measure-preserving transformation, for a large class of roof functions. We thus prove that the resulting random dynamical system satisfies the central, functional central and local limit theorems. As a corollary, it is rationally ergodic, with return sequence \sqrt{n} .

1. INTRODUCTION

Let (X, ν, T) be a probability measure-preserving system, and let λ be the Lebesgue measure on \mathbb{R} . Of course, the \mathbb{R} -extension $\Phi_0 : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ given by

$$\Phi_0(x, t) = (Tx, t),$$

which preserves the infinite product measure $\nu \times \lambda$, is not ergodic, as it has a Φ_0 -invariant foliation by horizontal circles.

Now let $\phi : X \rightarrow \mathbb{R}$ be a non-identically zero, measurable function with $\int_X \phi d\nu = 0$, and consider the \mathbb{R} -extension $\Phi_1 : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ defined by

$$\Phi_1(x, t) = (Tx, t + \phi(x)),$$

from now on called a *conservative cylinder flow*. Again, Φ_1 preserves $\nu \times \lambda$. There are clear obstructions for ergodicity. For example, if ϕ is a coboundary for T , then Φ_1 is conjugate to Φ_0 and thus is not ergodic. This shows that not only ϕ and T separately determine the ergodic properties of Φ_1 , but also the way ϕ relates to T .

In the present paper we study ergodic properties of random iterations of such conservative cylinder flows. Because the invariant foliations of Φ_0 and Φ_1 are different, it might occur that this random dynamical system, from now on called *random conservative cylinder flow*, is ergodic. The theorem below characterizes, in terms of ϕ , when this happens. As ergodicity is a measure-theoretical invariant, also must be the condition on ϕ . Define the *essential image* of ϕ to be the set of $t \in \mathbb{R}$ for which $\phi^{-1}[t - \varepsilon, t + \varepsilon]$ has positive ν -measure, for any $\varepsilon > 0$. We thus have the

Theorem 1.1. *Let (X, ν, T) be an invertible, ergodic probability measure-preserving system, and let $\phi : X \rightarrow \mathbb{R}$ be a measurable function with $\int_X \phi d\nu = 0$. Then the random conservative cylinder flow*

$$\begin{aligned} F & : \Omega_2 \times X \times \mathbb{R} &\longrightarrow & \Omega_2 \times X \times \mathbb{R} \\ & (\omega, x, t) &\longmapsto & (\sigma\omega, Tx, t + \omega_0\phi(x)) \end{aligned}$$

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is ergodic if and only if the closed subgroup generated by the essential image of ϕ is \mathbb{R} .

Above, $\Omega_k = \{0, 1, \dots, k-1\}^{\mathbb{Z}}$ denotes the space of two-sided sequences $\omega = (\omega_n)_{n \in \mathbb{Z}}$ of k symbols equipped with the uniform Bernoulli measure μ , $\sigma : \Omega_k \rightarrow \Omega_k$ is the *shift map*, which preserves the measure μ , and the measure considered on $\Omega_k \times X \times \mathbb{R}$ is the product measure $\mu \times \nu \times \lambda$, which is preserved under iterations of F . Indeed, from now on we always assume, without further explanation, that the product of measure-preserving transformations is equipped with the respective product measure.

On the point of view of abstract ergodic theory, Theorem 1.1 is a result on random dynamical systems with infinite measure, and it is a consequence of a more general statement about random conservative cylinder flows, which is the content of Theorem 1.2 below. Let us briefly describe the setup: (X, ν, T) is again an invertible, ergodic probability measure-preserving system, and $\phi_0, \dots, \phi_{k-1} : X \rightarrow \mathbb{R}$ are measurable functions, all of which with zero integral. For $i = 0, 1, \dots, k-1$, let $\Phi_i : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ be the conservative cylinder flow defined by

$$\Phi_i(x, t) = (Tx, t + \phi_i(x)),$$

which preserves the measure $\nu \times \lambda$. Let us say that the system $\{\Phi_0, \dots, \Phi_{k-1}\}$ is *ergodic* if a measurable subset $A \subset X \times \mathbb{R}$ such that

$$A = \Phi_0^{-1}A = \Phi_1^{-1}A = \dots = \Phi_{k-1}^{-1}A$$

either has zero or full measure. Alternatively, $\{\Phi_0, \dots, \Phi_{k-1}\}$ is ergodic if every bounded function $g : X \times \mathbb{R} \rightarrow \mathbb{R}$ which is invariant under $\Phi_0, \dots, \Phi_{k-1}$ is constant almost everywhere.

Theorem 1.2. *Let (X, ν, T) be an invertible, ergodic probability measure-preserving system, and let $\Phi_0, \dots, \Phi_{k-1} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ be conservative cylinder flows. Then the random conservative cylinder flow*

$$\begin{aligned} F & : \Omega_k \times X \times \mathbb{R} & \longrightarrow & \Omega_k \times X \times \mathbb{R} \\ & (\omega, x, t) & \longmapsto & (\sigma\omega, \Phi_{\omega_0}(x, t)) \end{aligned}$$

is ergodic if and only if the system $\{\Phi_0, \dots, \Phi_{k-1}\}$ is ergodic.

Theorem 1.2 is a version of Kakutani's random ergodic theorem (originally proved in [10]) for conservative cylinder flows. The first version of this theorem for infinite measure-preserving transformations appeared in a paper of Woś [17]. Even Theorem 1.2 may be concluded directly from [17], we provide the proof since the conditions that guarantee ergodicity for the random cylinder flow are much easier to formulate than that in [17] and similar to those usually used in infinite ergodic theory. We would like to thank David Sauzin for pointing out Woś' theorem. Indeed, he has a strong application of such result for the context of standard maps [13].

Yet on the point of view of abstract ergodic theory, it is known that classical theorems are no longer valid for infinite measures. For instance, Birkhoff's averages converge to zero almost surely, and this leads us to the following question: what would be a good candidate for a Birkhoff-type theorem in this context? One attempt of obtaining this has been made by Aaronson, who introduced the notion of *rational ergodicity* (see Subsection 2.2 for the proper definition). Denoting the Birkhoff sum of a function f by $S_n f$, rationally ergodic maps possess a sort of Cèsaro-averaged

version of convergence in measure: there is a sequence (a_n) such that, for every L^1 -function f and every subsequence (n_k) of positive integers, there exists a further subsequence (n_{k_l}) such that $S_{n_{k_l}} f(x)/a_{n_{k_l}}$ converges to $\int f$ almost everywhere. This latter property is called *weak homogeneity* and the sequence (a_n) , unique up to asymptotic equality, is called *return sequence*.

Many authors have investigated the ergodicity of specific classes of cylinder flows. See e.g. [7], [8], [11], [15], [16]. On the other hand, few examples of rationally ergodic cylinder flows are known: [3] studies the asymptotic behavior of random walks driven by irrational rotations on the circle, and shows that the corresponding cylinder flow is rationally ergodic with return sequence $a_n = n/\sqrt{\log n}$, and [6] constructs examples of cylinder flows over almost every irrational rotation on the circle that are rationally ergodic along a subsequence of iterates, with return subsequence $a_{q_n} = q_{n+1}/\sqrt{\pi n}$, where (q_n) is a subsequence of best approximations of the respective irrational rotation.

The second part of this work discusses these notions for a class of random conservative cylinder flows. Specifically, we reinforce the properties on (X, ν, T) and ϕ and then strength Theorem 1.1 by proving that F is also rationally ergodic, as stated below.

Theorem 1.3. *Let (X, ν, T) be an invertible, uniquely ergodic probability-measure preserving system, and let $\phi : X \rightarrow \mathbb{R}$ be a non-identically zero, continuous function such that*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi(T^i x) \rightarrow 0 \quad \text{uniformly in } x. \quad (1.1)$$

Then the random conservative cylinder flow

$$\begin{aligned} F &: \Omega_2 \times X \times \mathbb{R} \longrightarrow \Omega_2 \times X \times \mathbb{R} \\ (\omega, x, t) &\longmapsto (\sigma\omega, Tx, t + \omega_0\phi(x)) \end{aligned}$$

is rationally ergodic, with return sequence $a_n = \sqrt{n}$.

Theorem 1.3 holds, for example, when ϕ is a coboundary for T . Observe that assumption (1.1) is natural in the sense that, in order to obtain results such as the central and functional central limit theorems, the divergence of ϕ 's Birkhoff sums must be slower than that generated by the classical random walk on \mathbb{Z} .

The proof of Theorem 1.3 is based on a uniform local limit theorem with moving targets for independent but non-identically distributed Bernoulli random variables, on which the variances are driven by the values of ϕ along the orbits of T . More specifically, for each $x \in X$, observe that the third coordinate of $F^n(\omega, x, t)$ is equal to

$$t + \sum_{i=0}^{n-1} \omega_i \phi(T^i x) = t + \frac{1}{2} \sum_{i=0}^{n-1} \phi(T^i x) (2\omega_i - 1) + \frac{1}{2} \sum_{i=0}^{n-1} \phi(T^i x) \quad (1.2)$$

and so if we consider, for each $x \in X$, the martingale $(S_n^x)_{n \geq 1}$ defined by

$$S_n^x = \phi(x) \cdot X_0 + \phi(Tx) \cdot X_1 + \cdots + \phi(T^{n-1}x) \cdot X_{n-1}$$

where X_1, X_2, \dots are independent random variables, each distributed with

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = \frac{1}{2},$$

then (1.2) is equal to $t + \frac{1}{2}(S_n^x + \sum_{i=0}^{n-1} \phi(T^i x))$. Thus the behavior of the third coordinate of F^n is driven by the martingales $(S_n^x)_{n \geq 1}$, $x \in X$.

Observe that condition (1.1) and the mere fact that $(S_n^x)_{n \geq 1}$ is a martingale imply that the sequences $\sum_{i=0}^{n-1} \omega_i \phi(T^i x)$, $n \geq 1$, satisfy both the central and functional central limit theorems. Nevertheless, these results do not imply that F is rationally ergodic. For this we need a local limit theorem with moving target which is uniform in $x \in X$.

Theorem 1.4. *Assume the conditions of Theorem 1.3, and let (s_n^x) be sequences of real numbers such that*

$$\lim_{n \rightarrow \infty} s_n^x / \sqrt{n} = 0 \quad \text{uniformly in } x.$$

Then, for all $t > 0$, there are constants $K, n_0 > 0$ such that

$$K^{-1} \leq \sqrt{n} \cdot \mathbb{P}[S_n^x \in [-t, t] - s_n^x] \leq K$$

for every $n > n_0$ and $x \in X$.

The proof of Theorem 1.4 is contained in Section 5, and follows the steps of the Fourier analytical proof of classical local limit theorems. See for instance §5.2 of [2].

As a particular case of Theorem 1.1, let T be an irrational rotation on the circle, and let ϕ be a non-identically zero, continuous function with zero integral. If the values of ϕ are small, then Φ_1 can be seen as a conservative perturbation of Φ_0 , a particular situation that naturally appears in the phenomenon called *Arnold diffusion*. In [14] it has been proposed that a small perturbation in the Gevrey category of a non-degenerate integrable Hamiltonian system gives rise to a dynamics that can be reduced to a skew product extension of integrable cylinder flows over a shift of finite symbols, but having different rotations in each cylinder flow. Moreover, it is proved that the random trajectories of the random dynamical system converge to a Brownian motion, after a proper change of scales.

Our result can be applied to a slight variation of the model proposed in [14] (where the rotations are the same), in which case we also obtain a uniform local limit theorem with moving targets (Theorem 1.4) and rational ergodicity of the respective random cylinder flow (Theorem 1.3). We believe that these results can be generalized when different rotations appear in the different cylinder flows, which is closer to (but yet not the same as) the examples constructed in [14].

The paper is organized as follows. After establishing the necessary preliminaries, Section 3 is devoted to the proof of Theorem 1.2. We then use this result to prove Theorem 1.1 in Section 4. This completes the first part of the article. The second part is contained in Sections 5 and 6, and consists of the proofs of Theorems 1.4 and 1.3, respectively.

Remark 1.5. Theorems 1.1 to 1.4 hold for any Bernoulli measure on Ω_k , instead of only the uniform one. Indeed, Theorems 1.1 and 1.2 only make use that μ satisfies the Lebesgue differentiation theorem, while Theorems 1.3 and 1.4 only need that the random variables $\omega_0, \omega_1, \dots$ are independent and identically distributed.

2. NOTATION AND PRELIMINARIES

Definition 2.1. Let $f, g : \mathbb{N} \rightarrow \mathbb{R}$ be two real-valued functions. We say $f \lesssim g$ if there is a constant $C > 0$ such that

$$|f(n)| \leq C \cdot |g(n)|, \quad \forall n \in \mathbb{N}.$$

If $f \lesssim g$ and $g \lesssim f$, we write $f \sim g$.

An element of \mathbb{R} will be denoted by t , and λ the Lebesgue measure on \mathbb{R} . Ω_k is the space of two-sided sequences $\{0, 1, \dots, k-1\}^{\mathbb{Z}}$ with k symbols and $\omega = (\omega_n)_n$ will denote an arbitrary element of Ω_k .

The natural transformation on Ω_k is the left shift $\sigma : \Omega_k \rightarrow \Omega_k$, defined by the equality $(\sigma\omega)_n = \omega_{n+1}$. All measures on Ω_k will be Bernoulli measures, defined from a probability vector $\mathbf{p} = (p_0, \dots, p_{k-1})$ by the formula

$$\mu = \prod_{n \in \mathbb{Z}} \mathbf{p}.$$

Clearly, all such measures are invariant under σ .

2.1. Stable and unstable sets. Let $\Omega = \Omega_k$, and let

$$\Omega^s = \prod_{n < 0} \{0, 1\} \quad , \quad \Omega^u = \prod_{n \geq 0} \{0, 1\}.$$

Clearly $\Omega = \Omega^s \times \Omega^u$. Furthermore, if

$$\mu^s = \prod_{n < 0} \mathbf{p} \quad , \quad \mu^u = \prod_{n \geq 0} \mathbf{p}$$

are the Bernoulli measures on Ω^s, Ω^u respectively, then $\mu = \mu^s \times \mu^u$.

Ω can be seen either as a union of copies of Ω^s or of Ω^u . More specifically, define for $\omega \in \Omega$ the sets

$$\begin{aligned} \Omega^s[\omega] &= \{\omega' \in \Omega; \omega'_n = \omega_n \text{ for } n \geq 0\} \\ \Omega^u[\omega] &= \{\omega' \in \Omega; \omega'_n = \omega_n \text{ for } n < 0\} \end{aligned} ,$$

which are respectively the local stable and unstable sets of ω with respect to σ . It is clear that, for any $\omega \in \Omega$ one has

$$\bigcup_{\omega' \in \Omega^u[\omega]} \Omega^s[\omega'] = \Omega = \bigcup_{\omega' \in \Omega^s[\omega]} \Omega^u[\omega'].$$

Furthermore, $\Omega^s[\omega], \Omega^u[\omega]$ are naturally identified with Ω^s, Ω^u and thus can be equipped with the measures μ^s, μ^u respectively. This being said, for any $A \subset \Omega$ measurable and $\omega \in \Omega$, it holds that

$$\mu(A) = \int_{\Omega^u[\omega]} \mu^s(A \cap \Omega^s[\omega']) d\mu^u(\omega') = \int_{\Omega^s[\omega]} \mu^u(A \cap \Omega^u[\omega']) d\mu^s(\omega').$$

Observe that, even $(\Omega^u, \sigma, \mu^u)$ is a measure-preserving system, the measures $\mu^u \circ \sigma$ and μ^u are different (an analogous statement holds for μ^s). Nevertheless, both μ^s and μ^u have the bounded distortion property: for $\omega \in \Omega$, let $C_n^s[\omega] \subset \Omega^s[\omega]$ and $C_n^u[\omega] \subset \Omega^u[\omega]$ be defined by

$$\begin{aligned} C_n^s[\omega] &= \{\omega' \in \Omega; \omega'_k = \omega_k \text{ for } k > -n\} \\ C_n^u[\omega] &= \{\omega' \in \Omega; \omega'_k = \omega_k \text{ for } k < n\}. \end{aligned}$$

Lemma 2.2. *If $A, B \subset C_n^u[\omega]$, then*

$$\frac{\mu^u(\sigma^n A)}{\mu^u(\sigma^n B)} = \frac{\mu^u(A)}{\mu^u(B)}.$$

Analogously, if $A, B \subset C_n^s[\omega]$, then

$$\frac{\mu^s(\sigma^{-n} A)}{\mu^s(\sigma^{-n} B)} = \frac{\mu^u(A)}{\mu^u(B)}.$$

Proof. Let $I_j = [p_0 + \dots + p_{j-1}, p_0 + \dots + p_j]$, $j = 0, \dots, k-1$, and let $\Sigma : [0, 1) \rightarrow [0, 1)$ be the expanding piecewise linear map such that each $\Sigma|_{I_j}$ is linear onto $[0, 1)$. Define $\pi : \Omega^u \rightarrow [0, 1)$ by the equality

$$\{\pi(\omega)\} = \bigcap_{n \geq 0} \Sigma^{-n}(I_{\omega_n}).$$

Then $\pi \circ \sigma = \Sigma \circ \pi$. Furthermore, the push forward $\pi_* \mu^u$ is the Lebesgue measure $\lambda|_{[0,1]}$, and π is injective except at a set of zero μ^u -measure. Thus π is a measurable isomorphism between the probability measure-preserving systems $(\Omega^u, \mu^u, \sigma)$ and $([0, 1), \lambda|_{[0,1]}, \Sigma)$, and so

$$\frac{\mu^u(\sigma^n A)}{\mu^u(\sigma^n B)} = \frac{\lambda(\Sigma^n \pi A)}{\lambda(\Sigma^n \pi B)} = \frac{\lambda(\pi A)}{\lambda(\pi B)} = \frac{\mu^u(A)}{\mu^u(B)},$$

where in the second equality we used that $\Sigma^n|_{\pi(C_n^u[\omega])} : \pi(C_n^u[\omega]) \rightarrow [0, 1)$ is linear. The other statement is proved in an analogous manner. \square

Lemma 2.3. *Let $A \subset \Omega$ with $\mu(A) > 0$. If*

$$\Omega^s[\omega], \Omega^u[\omega] \subset A$$

for μ -almost every $\omega \in A$, then $A = \Omega$.

Proof. Let $\pi^s : \Omega \rightarrow \Omega^s$ and $\pi^u : \Omega \rightarrow \Omega^u$ be the canonical projections. Thus $\Omega^s[\omega] = \Omega^s \times \{\pi^u(\omega)\}$ and $\Omega^u[\omega] = \{\pi^s(\omega)\} \times \Omega^u$. Because $\Omega^s[\omega] \subset A$ for μ -almost every $\omega \in A$,

$$A = \bigcup_{\omega \in A} \Omega^s[\omega] = \bigcup_{\omega \in A} \Omega^s \times \{\pi^u(\omega)\} = \Omega^s \times \pi^u(A).$$

Analogously, $A = \pi^s(A) \times \Omega^u$, and thus

$$\Omega^s \times \pi^u(A) = \pi^s(A) \times \Omega^u \implies \pi^u(A) = \Omega^u \implies A = \Omega^s \times \Omega^u = \Omega.$$

\square

We end up this subsection establishing a Lebesgue differentiation theorem for μ^s and μ^u .

Lemma 2.4. *Let $A \subset \Omega^s[\omega]$. Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\mu^s(A \cap C_n^s[\omega'])}{\mu^s(C_n^s[\omega'])}$$

equals 1 for μ^s -almost every $\omega' \in A$ and 0 for μ^s -almost every $\omega' \notin A$. An analogous statement holds for $\Omega^u, \mu^u, C_n^u[\omega']$.

Proof. Let $\pi : \Omega^u[\omega] \rightarrow [0, 1)$ be as in the proof of Lemma 2.2. The set $\pi(C_n^u[\omega'])$ is an interval of length $\mu^u(C_n^u[\omega'])$ which contains $\pi(\omega')$. Because

$$\frac{\mu^u(A \cap C_n^u[\omega'])}{\mu^u(C_n^u[\omega'])} = \frac{\lambda(\pi(A) \cap \pi(C_n^u[\omega']))}{\lambda(\pi(C_n^u[\omega']))},$$

the Lebesgue differentiation theorem for λ gives the desired result. \square

2.2. Infinite ergodic theory. Let (Y, η, Φ) be an ergodic measure-preserving system, where η is a sigma-finite measure with $\eta(Y) = \infty$. Assume that Φ is *conservative*: $\eta(A) = 0$ for any measurable $A \subset Y$ such that $\{\Phi^{-n}A\}_{n \geq 0}$ are pairwise disjoint.

As stated in the introduction, for every $f \in L^1(Y, \eta)$ the Birkhoff averages $S_n f(x)/n$ converge to zero η -almost everywhere. Nevertheless, Hopf's ratio ergodic theorem is an indication that some sort of regularity might exist and it might still be possible, for a specific sequence (a_n) , to smooth out the fluctuations of $S_n f/a_n$ by means of a summability method.

One attempt of obtaining this has been made by Aaronson, who introduced the notion of rational ergodicity (see §3.3 of [2]). Given a measurable subset $A \subset Y$, let $R_n : A \rightarrow \mathbb{N}$ be the return function of A with respect to Φ :

$$R_n(y) = \#\{1 \leq i \leq n; \Phi^i(y) \in A\}.$$

Definition 2.5. A conservative ergodic measure-preserving system (Y, η, Φ) is called *rationally ergodic* if there is a measurable set $A \subset Y$ with $0 < \eta(A) < \infty$ such that the return function $R_n : A \rightarrow \mathbb{N}$ satisfies a *Renyi inequality*:

$$\int_A R_n^2 d\eta \lesssim \left(\int_A R_n d\eta \right)^2.$$

A theorem of Aaronson [1] (see also Theorem 3.3.1 of [2]) states that every rationally ergodic conservative measure-preserving system is *weakly homogeneous*. More specifically, it says that if

$$a_n = \frac{1}{\eta(A)^2} \int_A R_n d\eta = \frac{1}{\eta(A)^2} \sum_{i=1}^n \eta(A \cap \Phi^{-i}A), \quad (2.1)$$

then every subsequence (n_k) of positive integers can be refined to a further subsequence (n_{k_l}) such that for all $f \in L^1(Y, \eta)$ it holds that

$$\frac{1}{N} \sum_{l=1}^N \frac{1}{a_{n_{k_l}}} S_{n_{k_l}} f(y) \longrightarrow \int_Y f d\eta \quad \text{a.e.}$$

(a_n) is called the *return sequence* of Φ and it is unique up to asymptotic equality.

We conclude this subsection stating a result, due to Atkinson [4], which will be used in the next section.

Theorem 2.6. *Let (X, ν, T) be an invertible, ergodic probability measure-preserving system and $\phi : X \rightarrow \mathbb{R}$ a measurable function such that $\int_X \phi d\nu = 0$. Then ν -almost every $x \in X$ has the following property: for any measurable set $A \subset X$ containing x with $\nu(A) > 0$ and any $\varepsilon > 0$, the set*

$$\{n > 0; T^n x \in A \text{ and } |S_n \phi(x)| < \varepsilon\}$$

is infinite.

As a corollary, a cylinder flow $(x, t) \mapsto (Tx, t + \phi(x))$ is conservative whenever $\int_X \phi d\nu = 0$.

3. KAKUTANI'S THEOREM FOR CONSERVATIVE CYLINDER FLOWS

We remind that (X, ν, T) is an invertible, ergodic probability measure-preserving system, $\phi_0, \dots, \phi_{k-1} : X \rightarrow \mathbb{R}$ are measurable functions such that

$$\int_X \phi_0 d\nu = \dots = \int_X \phi_{k-1} d\nu = 0$$

and $\Phi_i(x, t) = (Tx, t + \phi_i(x))$, $i = 0, 1, \dots, k-1$, and let $\Omega = \Omega_k$. We want to prove that the random cylinder flow

$$\begin{aligned} F &: \Omega \times X \times \mathbb{R} \longrightarrow \Omega \times X \times \mathbb{R} \\ (\omega, x, t) &\longmapsto (\sigma\omega, \Phi_{\omega_0}(x, t)) \end{aligned}$$

is ergodic if and only if the system $\{\Phi_0, \dots, \Phi_{k-1}\}$ is ergodic.

Clearly, if F is ergodic then also is $\{\Phi_0, \dots, \Phi_{k-1}\}$. For instance, if $g(x, t)$ is invariant simultaneously for $\Phi_0, \dots, \Phi_{k-1}$, then $f(\omega, x, t) = g(x, t)$ is F -invariant.

The converse will follow from the

Lemma 3.1. *Any bounded F -invariant function $f(\omega, x, t)$ does not depend on the first coordinate, i.e. there is a bounded function $g(x, t)$ such that*

$$f(\omega, x, t) = g(x, t) \quad \text{a.e.}$$

Assume Lemma 3.1 has been proved and let $f(\omega, x, t) = g(x, t)$ be invariant under F . Whenever $\omega_0 = i$, this gives

$$(g \circ \Phi_i)(x, t) = f(\sigma\omega, \Phi_i(x, t)) = (f \circ F)(\omega, x, t) = f(\omega, x, t) = g(x, t)$$

and so g is invariant under each Φ_i . By assumption, g is constant almost everywhere and thus the same holds for f .

Proof of Lemma 3.1. Fix a set $A \subset \Omega \times X \times \mathbb{R}$ of positive measure, invariant under F . The idea is to see Ω as the union of either stable sets or unstable sets and restrict the action of σ to these subsets.

According to Lemma 2.3, it is enough to prove that

$$\Omega^s[\omega] \times \{(x, t)\} \quad \text{and} \quad \Omega^u[\omega] \times \{(x, t)\} \subset A \tag{3.1}$$

for almost every $(\omega, x, t) \in A$. Let $A^{(x, t)} \subset \Omega$ be defined by the equality

$$A = \bigcup_{(x, t) \in X \times \mathbb{R}} A^{(x, t)} \times \{(x, t)\}$$

and define measurable functions $f_1, f_2, \dots : A \rightarrow \mathbb{R}$ by

$$f_n(\omega, x, t) = \frac{\mu^u(C_n^u[\omega] \cap A^{(x, t)})}{\mu^u(C_n^u[\omega])}.$$

By Theorem 2.4, we have

$$\lim_{n \rightarrow \infty} f_n(\omega, x, t) = 1 \quad \text{for a.e. } (\omega, x, t) \in A. \tag{3.2}$$

Assume first that (3.2) holds uniformly in A . Fix $\delta > 0$ and let $n_0 \geq 1$ for which $f_n > 1 - \delta$ for all $n > n_0$.

By Theorem 2.6, for almost every $(\omega, x, t) \in A$ there is $n > n_0$ such that $(\omega', x', t') = F^{-n}(\omega, x, t)$ belongs to A , and then

$$\frac{\mu^u \left(C_n^u[\omega'] \cap A^{(x', t')} \right)}{\mu^u(C_n^u[\omega'])} > 1 - \delta. \quad (3.3)$$

Note that

$$\begin{cases} \omega' &= \sigma^{-n}\omega \\ x' &= T^{-n}x \\ t' &= t + S_{-n}\Phi(\omega, x) \end{cases}$$

where

$$S_{-n}\Phi(\omega, x) = - \sum_{j=1}^n \phi_{\omega_{-j}}(T^{-j}x).$$

Because $S_n\Phi$ only depends on x and on the first n coordinates of ω , we have

$$S_n\Phi|_{C_n^u[\omega'] \times \{x'\}} = S_n\Phi(\omega', x') = -S_{-n}\Phi(\omega, x) = t - t'.$$

Furthermore, $\sigma^n C_n^u[\omega'] = \Omega^u[\omega]$ and so

$$F^n(C_n^u[\omega'] \times \{(x', t')\}) = \Omega^u[\omega] \times \{(x, t)\}.$$

Thus Lemma 2.2 and relation (3.3) imply that

$$\begin{aligned} \mu^u \left(\Omega^u[\omega] \cap A^{(x, t)} \right) &= \frac{\mu^u \left(\sigma^n \left(C_n^u[\omega'] \cap A^{(x', t')} \right) \right)}{\mu^u(\sigma^n(C_n^u[\omega']))} \\ &= \frac{\mu^u \left(C_n^u[\omega'] \cap A^{(x', t')} \right)}{\mu^u(C_n^u[\omega'])} \\ &> 1 - \delta. \end{aligned}$$

Because both $(\omega, x, t) \in A$ and $\delta > 0$ are arbitrary, it follows that $\Omega^u[\omega] \subset A^{(x, t)}$ for almost every $(\omega, x, t) \in A$. Analogously, we also have that $\Omega^s[\omega] \subset A_{x, t}$ for almost every $(\omega, x, t) \in A$, and this establishes (3.1).

In the general situation, the convergence in (3.2) is not uniform. Instead, do the following: for each $A' \subset A$ with finite measure and each $\varepsilon > 0$, Egorov's theorem assures the existence of $A'' \subset A'$ such that

- (1) $(\mu \times \nu \times \lambda)(A'') > (\mu \times \nu \times \lambda)(A') - \varepsilon$, and
- (2) f_n converges to 1 uniformly in A'' .

By the previous argument, (3.1) hold almost everywhere in A'' . This concludes the proof. \square

As remarked in the introduction, Theorem 1.2 is a version of Kakutani's random ergodic theorem for conservative cylinder flows. We think Kakutani's random ergodic theorem should also hold more generally for any finite number of conservative, infinite measure-preserving transformations. Indeed, the argument presented here establishes this if one succeeds to prove that a random dynamical system generated by conservative, infinite measure-preserving transformations is conservative.

Note that conservativity is indeed a necessary condition. For example, take $\phi_0 = 0$ and ϕ_1 without zero integral such that the closed subgroup generated by the essential image of ϕ_1 is \mathbb{R} . As will be proved in the next section, this implies that the system $\{\Phi_0, \Phi_1\}$ is ergodic (equivalently, one can conclude this by invoking a

similar argument to that in [12]). On the other hand F is, by Theorem 2.6, a non-conservative measure-preserving transformation. If F was also ergodic, it would have to be isomorphic to the translation $n \mapsto n + 1$ on the integers¹, but one can choose ϕ_1 properly in such a way that this is not the case.

4. PROOF OF THEOREM 1.1

Let G be the closed subgroup generated by the essential image of ϕ . G is either equal to $\alpha\mathbb{Z}$ or \mathbb{R} . Assume $G = \alpha\mathbb{Z}$. If $\alpha = 0$, then $F(\omega, x, t) = (\sigma\omega, Tx, t)$ is clearly non-ergodic. If $\alpha \neq 0$, then

$$A = \Omega \times X \times (\alpha\mathbb{Z} + [0, \alpha/4])$$

is a non-trivial F -invariant set, so again F is non-ergodic.

Now assume $G = \mathbb{R}$. We want to prove that F is ergodic. Let $\Phi_0, \Phi_1 : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ be

$$\Phi_0(x, t) = (Tx, t) \quad \text{and} \quad \Phi_1(x, t) = (Tx, t + \phi(x)).$$

By Theorem 1.2, it is enough to prove that the system $\{\Phi_0, \Phi_1\}$ is ergodic. To this purpose, let $g(x, t)$ be a bounded function, invariant under Φ_0 and Φ_1 . Then

$$g(Tx, t) = (g \circ \Phi_0)(x, t) = g(x, t)$$

and so, by the ergodicity of T , g does not depend on the first coordinate, i.e. there is $h(t)$ such that $g(x, t) = h(t)$ almost everywhere.

It remains to prove that h is constant almost everywhere. Note that

$$h(t + \phi(x)) = g(Tx, t + \phi(x)) = (g \circ \Phi_1)(x, t) = g(x, t) = h(t)$$

and so $h(t + \phi(x)) = h(t)$ for almost every $t \in \mathbb{R}$ and $x \in X$. This means that the set

$$\mathcal{P} = \{s \in \mathbb{R}; h(t + s) = h(t) \text{ for almost every } t \in \mathbb{R}\}$$

contains the essential image of ϕ . Because \mathcal{P} is a closed subgroup², it is equal to \mathbb{R} , which in other words means that h , and thus also f , is constant almost everywhere. This concludes the proof.

5. A LOCAL LIMIT THEOREM: PROOF OF THEOREM 1.4

The aim of this section is to prove Theorem 1.4. To simplify the notation, denote $c_i^x = \phi(T^i x)$ and

$$s_n^x = \sum_{i=0}^{n-1} \phi(T^i x) = \sum_{i=0}^{n-1} c_i^x.$$

¹An invertible, non-conservative, ergodic measure-preserving transformation is, up to isomorphism, equal to the translation $n \mapsto n + 1$ on the integers. See Proposition 1.2.1 of [2].

²It is clear that \mathcal{P} is a subgroup. To prove that \mathcal{P} is closed, note that it can also be characterized, by the Riesz representation theorem, as

$$\mathcal{P} = \left\{ s \in \mathbb{R}; \int_{\mathbb{R}} h(t + s)u(t)dt = \int_{\mathbb{R}} h(t)u(t)dt \text{ for every } u \in C_c(\mathbb{R}) \right\},$$

where $C_c(\mathbb{R})$ is the set of continuous functions $u : \mathbb{R} \rightarrow \mathbb{R}$ of compact support. By the dominated convergence theorem, \mathcal{P} is closed.

The third coordinate of $F^n(\omega, x, t)$, being equal to

$$t + \sum_{i=0}^{n-1} \omega_i \cdot c_i^x = t + \frac{1}{2} \sum_{i=0}^{n-1} c_i^x \cdot X_i + \frac{s_n^x}{2},$$

is ruled out by $S_n^x : \Omega \rightarrow \mathbb{R}$ given as

$$S_n^x = \sum_{i=0}^{n-1} c_i^x \cdot X_i,$$

where X_1, X_2, \dots are independent random variables each with law

$$X_0 = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases}. \quad (5.1)$$

For a fixed $x \in X$, the process S_1^x, S_2^x, \dots is a martingale with bounded increments and thus, by the central limit theorem for martingales with bounded increments, it satisfies the central limit theorem, and indeed the functional central limit theorem [9]. This means that the orbits under F distribute in a normal regime. It can also be shown, using the classical Fourier analytical proof of the local limit theorem (see e.g. §10.4 of [5]) together with the fact that the orbit of x under T equidistributes on X , that each of these processes also satisfies the local limit theorem

$$\sqrt{2\pi n} \cdot \mathbb{P}[S_n^x \in [a, b]] \rightarrow b - a \quad \text{as } n \rightarrow \infty$$

for any $x \in X$, and even that

$$\sqrt{2\pi n} \cdot \mathbb{P}[S_n^x \in [a, b] - s_n] \rightarrow b - a \quad \text{as } n \rightarrow \infty$$

for any $x \in X$, and any sequence (s_n) such that $s_n/\sqrt{n} \rightarrow 0$. This is not enough for proving rational ergodicity, because different x 's may give different rates of convergence above. As rational ergodicity does not take into account multiplicative constants, we only need a weaker, but uniform on x and n , statement. This is the content of Theorem 1.4, which we'll now prove.

From now on we assume, after a proper dilation, that $t = 1$. The proof proceeds as follows: firstly, we use the unique ergodicity of (X, ν, T) to obtain an estimate of the characteristic function of S_n^x , uniform on n and x . Secondly, we use such estimate, together with a Fourier analytical approach, to establish the result.

Given a random variable Y , let $\varphi_Y : \mathbb{R} \rightarrow \mathbb{C}$ denote its characteristics function:

$$\varphi_Y(t) = \mathbb{E}[\exp(itY)].$$

Lemma 5.1. *There exist $\delta, a, b, n_0 > 0$ such that for every $n > n_0$ and $x \in X$, it holds that*

$$\exp(-at^2) \leq \varphi_{S_n^x} \left(\frac{t}{\sqrt{n}} \right) \leq \exp(-bt^2), \quad \forall |t| \leq \delta\sqrt{n}.$$

Proof. Note that

$$\varphi_{X_0}(t) = \cos t = 1 - \frac{t^2}{2} + O(t^4)$$

and so, for $|t|$ small,

$$\log \varphi_{X_0}(t) \leq \log \left(1 - \frac{t^2}{4}\right) \leq -\frac{t^2}{8}$$

and

$$\log \varphi_{X_0}(t) \geq \log(1 - t^2) \geq -2t^2.$$

Thus, letting $C = \|\phi\|_\infty > 0$, take $\delta > 0$ small so that

$$\exp(-2t^2) \leq \varphi_{X_0}(t) \leq \exp(-t^2/8), \quad \forall |t| < \delta C. \quad (5.2)$$

Because

$$\varphi_{S_n^x} \left(\frac{t}{\sqrt{n}} \right) = \varphi_{X_0} \left(\frac{c_0^x t}{\sqrt{n}} \right) \cdots \varphi_{X_0} \left(\frac{c_{n-1}^x t}{\sqrt{n}} \right),$$

(5.2) implies that, for every $|t| < \delta\sqrt{n}$,

$$\exp \left(-\frac{2 \sum_{i=0}^{n-1} (c_i^x)^2}{n} \cdot t^2 \right) \leq \varphi_{S_n^x} \left(\frac{t}{\sqrt{n}} \right) \leq \exp \left(-\frac{\sum_{i=0}^{n-1} (c_i^x)^2}{8n} \cdot t^2 \right).$$

By Birkhoff's ergodic theorem, there is $n_0 > 0$ such that

$$\frac{1}{2} \int \phi^2 d\nu \leq \frac{1}{n} \sum_{i=0}^{n-1} (c_i^x)^2 \leq 2 \int \phi^2 d\nu, \quad \forall n \geq n_0, \forall x \in X.$$

Take

$$a = 4 \int \phi^2 d\nu \quad \text{and} \quad b = \frac{1}{16} \int \phi^2 d\nu$$

to conclude the proof of the lemma. \square

Let $\chi_{[-1,1]}$ denote the indicator function of the interval $[-1, 1]$. For the proof of Theorem 1.4, fix functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ such that

- (i) $g \leq \chi_{[-1,1]} \leq h$,
- (ii) $\hat{g}(0) > 0$, and
- (iii) \hat{g}, \hat{h} are continuous with support contained in $[-\Delta, \Delta]$, for some $\Delta > 0$.

It is not hard to see that such functions exist. One can take, for example,

$$g = \frac{1}{12} \left[\left(\frac{\widehat{\chi_{[-4,4]}}}{4} \right)^4 - \left(\frac{\widehat{\chi_{[-4,4]}}}{4} \right)^2 \right] \quad \text{and} \quad h = \widehat{\chi_{[-1,1]}}^2.$$

We can also assume, by conditions (ii) and (iii) above, that $\delta > 0$ also satisfies

- (iv) $\hat{g}|_{[-\delta, \delta]} > \hat{g}(0)/2$.

Proof of Theorem 1.4. We want to estimate

$$\sqrt{n} \cdot \mathbb{P}[S_n^x \in [-1, 1] - s_n^x] = \sqrt{n} \cdot \mathbb{E}[\chi_{[-1,1]}(S_n^x + s_n^x)].$$

Because

$$\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)] \leq \sqrt{n} \cdot \mathbb{E}[\chi_{[-1,1]}(S_n^x + s_n^x)] \leq \sqrt{n} \cdot \mathbb{E}[h(S_n^x + s_n^x)],$$

it is enough to estimate $\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)]$ from below and $\sqrt{n} \cdot \mathbb{E}[h(S_n^x + s_n^x)]$ from above.

Part 1. Bound of $\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)]$ from below.

By the Fourier inverse theorem,

$$\begin{aligned}
\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)] &= \sqrt{n} \cdot \mathbb{E} \left[\int_{\mathbb{R}} \hat{g}(t) \exp(it(S_n^x + s_n^x)) dt \right] \\
&= \sqrt{n} \cdot \int_{\mathbb{R}} \hat{g}(t) \mathbb{E}[\exp(it(S_n^x + s_n^x))] dt \\
&= \sqrt{n} \cdot \int_{\mathbb{R}} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt \\
&= \sqrt{n} \cdot \int_{-\delta}^{\delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt + \sqrt{n} \cdot \int_{\delta < |t| < \Delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt.
\end{aligned}$$

We claim there is $\lambda < 1$ such that, for n_0 large enough, it holds that

$$|\varphi_{S_n^x + s_n^x}(t)| \leq \lambda^n, \quad \forall x \in X, \forall n > n_0, \forall \delta < |t| < \Delta. \quad (5.3)$$

To prove this, take $\varepsilon > 0$ sufficiently small such that $[\varepsilon, 2\varepsilon] \subset \phi(X)$ and

$$|\cos(st)| < \rho, \quad \forall s \in [\varepsilon, 2\varepsilon], \forall \delta < |t| < \Delta,$$

for some $\rho < 1$. Because ϕ is continuous and (X, ν, T) is uniquely ergodic, there is $n_0 > 0$ such that

$$\frac{\#\{0 \leq i < n; T^i x \in \phi^{-1}[\varepsilon, 2\varepsilon]\}}{n} > \alpha, \quad \forall x \in X, \forall n > n_0,$$

where $2\alpha = \nu(\phi^{-1}[\varepsilon, 2\varepsilon]) > 0$. Thus, for every $x \in X, n > n_0$ and $\delta < |t| < \Delta$, we have

$$\begin{aligned}
|\varphi_{S_n^x + s_n^x}(t)| &= |\varphi_{S_n^x}(t)| \\
&= |\cos(c_0^x t) \cdots \cos(c_{n-1}^x t)| \\
&\leq \prod_{\substack{0 \leq i < n \\ c_i^x \in [\varepsilon, 2\varepsilon]}} |\cos(c_i^x t)| \\
&< \rho^{\#\{0 \leq i < n; c_i^x \in [\varepsilon, 2\varepsilon]\}} \\
&= \rho^{\#\{0 \leq i < n; T^i x \in \phi^{-1}[\varepsilon, 2\varepsilon]\}} \\
&< \rho^{\alpha n} \\
&= \lambda^n,
\end{aligned}$$

where $\lambda = \rho^\alpha < 1$. This establishes (5.3).

Then

$$\left| \sqrt{n} \cdot \int_{\delta < |t| < \Delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt \right| < 2\Delta \|\hat{g}\|_\infty \cdot \sqrt{n} \cdot \lambda^n. \quad (5.4)$$

To estimate the integral close to zero, first apply a change of variables to get

$$\begin{aligned}
\sqrt{n} \cdot \int_{-\delta}^{\delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt &= \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \hat{g}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x + s_n^x}\left(\frac{t}{\sqrt{n}}\right) dt \\
&= \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \hat{g}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x}\left(\frac{t}{\sqrt{n}}\right) \exp\left(it \frac{s_n^x}{\sqrt{n}}\right) dt \\
&= \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \hat{g}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x}\left(\frac{t}{\sqrt{n}}\right) \cos\left(t \frac{s_n^x}{\sqrt{n}}\right) dt.
\end{aligned}$$

Now, let $\beta > 0$ such that $\cos|_{[-\beta, \beta]} > 1/2$, let

$$m_n^x = \min \left\{ \frac{\beta\sqrt{n}}{s_n^x}, \frac{\delta\sqrt{n}}{2} \right\},$$

and divide the former integral into two parts accordingly to m_n^x :

$$\begin{aligned} \sqrt{n} \cdot \int_{-\delta}^{\delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt &= \int_{|t| < m_n^x} \hat{g}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x}\left(\frac{t}{\sqrt{n}}\right) \cos\left(t \frac{s_n^x}{\sqrt{n}}\right) dt + \\ &\quad \int_{m_n^x < |t| < \delta\sqrt{n}} \hat{g}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x}\left(\frac{t}{\sqrt{n}}\right) \cos\left(t \frac{s_n^x}{\sqrt{n}}\right) dt \\ &= I_1 + I_2. \end{aligned}$$

By the choice of g and β , the fact that $m_n^x \rightarrow \infty$ as $n \rightarrow \infty$, and Lemma 5.1, it follows that

$$I_1 \geq \frac{\hat{g}(0)}{4} \int_{|t| < m_n^x} \exp(-at^2) dt \geq \frac{\hat{g}(0)}{4} \int_{-1}^1 \exp(-at^2) dt$$

for every sufficiently large n and arbitrary x . Now, by the same reasons as in the last estimative,

$$\begin{aligned} I_2 &\geq - \int_{m_n^x < |t| < \delta\sqrt{n}} \left| \hat{g}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x}\left(\frac{t}{\sqrt{n}}\right) \cos\left(it \frac{s_n^x}{\sqrt{n}}\right) \right| dt \\ &\geq - \int_{m_n^x < |t| < \delta\sqrt{n}} \|\hat{g}\|_{\infty} \exp(-bt^2) dt \\ &\geq - \|\hat{g}\|_{\infty} \int_{|t| > m_n^x} \exp(-bt^2) dt. \end{aligned}$$

Thus

$$\sqrt{n} \cdot \int_{-\delta}^{\delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt \geq \frac{\hat{g}(0)}{4} \int_{-1}^1 \exp(-at^2) dt - \|\hat{g}\|_{\infty} \int_{|t| > m_n^x} \exp(-bt^2) dt$$

is bounded away from zero if n is large, uniformly in x . This, together with (5.4), proves Part 1.

Part 2. Bound of $\sqrt{n} \cdot \mathbb{E}[h(S_n^x + s_n^x)]$ from above.

Analogously as in Part 1, inequality (5.3) gives

$$\left| \sqrt{n} \cdot \int_{\delta < |t| < \Delta} \hat{h}(t) \varphi_{S_n^x + s_n^x}(t) dt \right| < 2\Delta \|\hat{h}\|_{\infty} \cdot \sqrt{n} \cdot \lambda^n$$

and Lemma 5.1 gives

$$\begin{aligned}
\sqrt{n} \cdot \int_{-\delta}^{\delta} \hat{h}(t) \varphi_{S_n^x + s_n^x}(t) dt &= \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \hat{h}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x + s_n^x}\left(\frac{t}{\sqrt{n}}\right) dt \\
&\leq \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \hat{h}\left(\frac{t}{\sqrt{n}}\right) \varphi_{S_n^x}\left(\frac{t}{\sqrt{n}}\right) dt \\
&\leq \|\hat{h}\|_{\infty} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \exp(-bt^2) dt \\
&\leq \|\hat{h}\|_{\infty} \int_{\mathbb{R}} \exp(-bt^2) dt,
\end{aligned}$$

which is finite. \square

Observe that the above proof is robust in the following sense: Theorem 1.4 remains valid with the same constants K, n_0 if we change (s_n^x) by $(s_n^x + t)$, where t runs over a compact set.

6. RATIONAL ERGODICITY: PROOF OF THEOREM 1.3

We now proceed to prove rational ergodicity. It is enough to prove that if $R_n : \Omega \times X \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{N}$ is the return function,

$$\begin{aligned}
R_n(\omega, x, t) &= \# \left\{ 1 \leq i \leq n ; F^i(\omega, x, t) \in \Omega \times X \times \left[-\frac{1}{2}, \frac{1}{2}\right] \right\} \\
&= \sum_{i=1}^n \chi_{[-1,1]}(S_i^x(\omega) + s_i^x + 2t),
\end{aligned}$$

then

$$\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n(\omega, x, t)^2 \lesssim \left(\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n(\omega, x, t) \right)^2, \quad n > 0. \quad (6.1)$$

For fixed $x \in X$ and $t \in [-\frac{1}{2}, \frac{1}{2}]$, Theorem 1.4 gives that

$$\begin{aligned}
\int_{\Omega} R_n(\omega, x, t) &= \sum_{i=1}^n \int_{\Omega} \chi_{[-1,1]}(S_i^x(\omega) + s_i^x + 2t) \\
&= \sum_{i=1}^n \mathbb{P}[S_i^x \in [-1, 1] - (s_i^x + 2t)] \\
&\sim \sum_{i=n_0+1}^n \mathbb{P}[S_i^x \in [-1, 1] - (s_i^x + 2t)] \\
&\sim \sum_{n_0 < i \leq n} i^{-1/2} \\
&\sim \int_1^n x^{-1/2} dx \\
&\sim \sqrt{n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\Omega} R_n(\omega, x, t)^2 &= \sum_{i=1}^n \int_{\Omega} \chi_{[-1,1]}(S_i^x(\omega) + s_i^x + 2t) \\
&\quad + 2 \sum_{i < j} \int_{\Omega} \chi_{[-1,1]}(S_i^x(\omega) + s_i^x + 2t) \cdot \chi_{[-1,1]}(S_j^x(\omega) + s_j^x + 2t) \\
&= \int_{\Omega} R_n(\omega, x, t) \\
&\quad + 2 \sum_{i < j} \mathbb{P}[S_i^x \in [-1, 1] - (s_i^x + 2t), S_j^x \in [-1, 1] - (s_j^x + 2t)] \\
&\sim \sqrt{n} + \sum_{i < j} \mathbb{P}[S_i^x \in [-1, 1] - (s_i^x + 2t), S_j^x \in [-1, 1] - (s_j^x + 2t)].
\end{aligned}$$

By the cocycle property, we have

$$S_j^x(\omega) = S_i^x(\omega) + S_{j-i}^{T^i x}(\sigma^i \omega)$$

and so, because $S_i^x(\omega)$ and $S_{j-i}^{T^i x}(\sigma^i \omega)$ are independent³ in ω , it follows that whenever $i > n_0$ and $j - i > n_0$,

$$\begin{aligned}
\mathbb{P} \left[\begin{array}{c} S_i^x \in [-1, 1] - (s_i^x + 2t), \\ S_j^x \in [-1, 1] - (s_j^x + 2t) \end{array} \right] &\leq \mathbb{P} \left[\begin{array}{c} S_i^x \in [-1, 1] - (s_i^x + 2t), \\ S_{j-i}^{T^i x} \in [-2, 2] - (s_i^x + s_j^x) \end{array} \right] \\
&= \mathbb{P}[S_i^x \in [-1, 1] - (s_i^x + 2t)] \times \\
&\quad \mathbb{P}[S_{j-i}^{T^i x} \in [-2, 2] - (s_i^x + s_j^x)] \\
&\lesssim i^{-1/2} \cdot (j - i)^{-1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{i < j} \mathbb{P} \left[\begin{array}{c} S_i^x \in [-1, 1] - (s_i^x + 2t), \\ S_j^x \in [-1, 1] - (s_j^x + 2t) \end{array} \right] &\lesssim \sum_{\substack{1 \leq i < j \leq n \\ i, j - i > n_0}} i^{-1/2} \cdot (j - i)^{-1/2} \\
&\leq \left(\sum_{i=1}^n i^{-1/2} \right)^2 \\
&\sim n.
\end{aligned}$$

This gives that

$$\int_{\Omega} R_n(\omega, x, t)^2 \lesssim n$$

for any $x \in X$ and any $t \in [-\frac{1}{2}, \frac{1}{2}]$, and so

$$\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n(\omega, x, t)^2 \lesssim n \lesssim \left(\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n(\omega, x, t) \right)^2,$$

which concludes the proof of (6.1).

³ $S_i^x(\omega)$ depends on the coordinates $\omega_0, \dots, \omega_{i-1}$ and $S_{j-i}^{T^{i+1}x}(\sigma^{i+1}\omega)$ depends on the coordinates $\omega_i, \dots, \omega_{j-1}$.

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